1 Introduction

In a recent paper, we have introduced a new treatment of systems of compounded angular momentum which leads to very generalized formulas for the states and operators for such systems [1]. In the paper, we worked these quantities out explicitly for the cases of spin 0 and spin 1 resulting from the addition of two spins of 1/2 each. We first obtained the generalized probability amplitudes describing results of measurements on such systems, then used these to derive the matrix treatment of the systems. The forms that we obtained for the vector states and operators proved to be entirely different from the standard forms. However, as might be expected, the results of calculations of measurable quantities are the same. Nevertheless, the question how the vectors and operators belonging to the standard treatment of spin addition are related to the new forms requires an answer. Indeed, since we consider the treatment of spin addition by means of probability amplitudes as being the foundation of any matrix treatment, it is necessary to derive the standard quantities by the new approach.

In this paper, we demonstrate that the standard matrix treatment of compounded spin is indeed derivable by our method. The use of this approach not only yields the standard treatment, but also produces results more generalized than any in the literature. However, these results reduce to the standard forms in an appropriate limit. The systems on which the theory developed is tested are the triplet and singlet states resulting from the addition of the spins of two spin-1/2 systems. It is clear from the application of the theory to these cases how the extension to arbitrary systems of compounded spin is achieved.

The organization of the paper is as follows. After the introduction in Section 1, we give in Section 2 a brief description of those features of the Landé approach to quantum mechanics that we shall use to develop our treatment. In Section 3, we turn our attention to a review of the work we have so far done on systems of compounded angular momentum. We remind ourselves of the expressions for the probability amplitudes for the addition of general angular momentum in Section 3.1, and of the probability amplitudes for spin addition in Section 3.2.

In Section 4, we look at the way the transformation from wave or probability-amplitude mechanics to matrix mechanics is achieved. We sketch in Section 4.1 the derivation of matrix mechanics from probability-amplitudes mechanics for simple systems. In Section 4.2, we derive the standard form of matrix mechanics for systems of compounded spin - however, the new results are more generalized than the standard ones.

The results of Section 4 are employed on actual systems in Section 5. The test systems are the singlet and triplet states arising from the addition of the spins of two spin -1/2 systems. The matrix operator is common to both cases, and is calculated in Section 5.1. The vectors states are obtained in Section 5.2.

The results obtained in Section 5 are more generalized than the standard forms found in the literature. In Section 6 we demonstrate how to reduce these results to the standard forms.

We end the paper with a Discussion and Conclusion in Section 7.

2 Basic Theory

2.1 The Landé Approach to Quantum Mechanics

The basic theory underlying our work derives from the interpretation of quantum mechanics due to Landé [2-5]. Among many features of the Landé approach is the assumption that wave functions and eigenfunctions in quantum mechanics are

probability amplitudes. Any probability amplitude connects two states - one state pertaining to the situation that obtains before a measurement is made, and the other to the state that results from the measurement. Thus, an energy eigenfunction $\phi_E(\mathbf{r})$ resulting from solution of the time-independent Schrödinger equation is a probability amplitude connecting two states: the state defined by the eigenvalue E is the initial state, while the final state is characterized by the position eigenvalue $\mathbf{r}$. Thus $|\phi_E(\mathbf{r})|^2\,d\mathbf{r}$ is the probability that if the system is initially in the state corresponding to E, a measurement of the position gives $\mathbf{r}$ in the volume element $d\mathbf{r}$.

The Landé interpretation of quantum mechanics is based on the principle that nature is ultimately indeterministic and should be described by a theory that is fundamentally probabilistic. Consequently, the description of measurements can only be given in probabilistic terms through probability amplitudes. For a particular system, the different sets of probability amplitudes connecting different measurable quantities are inter-related in the following way.

Let a quantum system have the observables A, B and C which have the respective eigenvalue spectra A_1, A_2,...,A_N, B_1, B_2,...,B_N and C_1, C_2,..,C_N. If the system is initially in the state corresponding to the eigenvalue A_i, a measurement of B yields any of the eigenvalues B_j with probabilities determined by the probability amplitudes $\eta(A_i; B_j)$. A measurement of C results in one of the eigenvalues C_j with probabilities determined by the probability amplitudes $\psi(A_i; C_j)$. If the system is initially in the state corresponding to the eigenvalue B_i, a measurement of C gives any of the eigenvalues C_j with probabilities determined by the probability amplitudes $\xi(B_i; C_j)$. The probability amplitudes display a two-way symmetry contained in the Hermiticity condition

$$\psi(C_j; A_i) = \psi^*(A_i; C_j). \tag{1}$$

These probability amplitudes are orthogonal:

$$\sum_{j=1}^{N} \psi^*(A_i; C_j)\psi(A_k; C_j) = \delta_{ik}. \tag{2}$$

The law that connects the three sets of probability amplitudes is

$$\psi(A_i; C_n) = \sum_{j=1}^{N} \eta(A_i; B_j)\xi(B_j; C_n). \tag{3}$$

Though the features of the Landé approach highlighted above refer to probability amplitudes that correspond to a discrete final eigenvalue spectrum, there is no essential difference if this spectrum is instead continuous. In fact, if the observable C is the position $\mathbf{r}$, and if, as is customary, we ignore the initial state A_i in the labelling, Eq. (3) becomes

$$\psi(\mathbf{r}) = \sum_{j=1}^{N} \eta_j \xi_j(\mathbf{r}), \tag{4}$$

where we have set $\eta_j = \eta(B_j)$ and $\xi_j = \xi(B_j; C_n)$. We recognize this equation as the law of interference of probabilities. In the Landé formalism, this important relation is derived, not assumed.

The relation Eq. (4) is, of course, the basis for the transformation of representation from wave to matrix mechanics for the case where the eigenfunctions $\xi_j(\mathbf{r})$ are known from solution of some eigenvalue equation. By the same token, its parent relation Eq. (3) is the basis for the transformation of representation from probability-amplitude mechanics to matrix mechanics in all cases, irrespective of

whether or not a differential eigenvalue equation exists for the probability amplitudes $\xi(B_j; C_n)$. Indeed, this is the relation on which we have based the derivation of the matrix theory of spin from probability amplitudes [1,7-11].

3 Review of Previous Results on Angular Momentum Addition

3.1 General Theory

In a previous paper [1], we derived a matrix treatment of spin addition which resulted in new forms for the vectors and the operators, apart from throwing light on the theory of angular momentum addition. In this section, we review the new treatment of spin addition in order to present those results which will be needed in the development of the present work.

We consider first the case of general angular momentum addition. Let a system have the total angular momentum $\mathbf{J}$ resulting from adding the angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$ of subsystems 1 and 2. Thus,

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2. \tag{5}$$

The quantum numbers of the angular momenta of the subsystems are j_1 and j_2, while that of the angular momentum of the total system is j. The z components of these respective angular momenta are characterized by the quantum numbers m_1, m_2 and M. For the time being, we shall assume that $\mathbf{J}$, $\mathbf{J}_1$ and $\mathbf{J}_2$ are orbital angular momenta. The subsystems 1 and 2 are characterized by the angular variables (θ_1, φ_1) and (θ_2, φ_2) respectively. The standard expression for the wave function of the coupled system is

$$\Psi_{j_1 j_2 j M}(\theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1} C(j_1 j_2 j; m_1 m_2 M) \phi^{(1)}_{j_1 m_1}(\theta_1, \varphi_1) \phi^{(2)}_{j_2 m_2}(\theta_2, \varphi_2), \tag{6}$$

where we have used the notation in Rose [6] for the Clebsch-Gordan coefficients $C(j_1 j_2 j; m_1 m_2 M)$. If we are dealing with orbital angular momentum, the $\phi^{(1)}_{j_1 m_1}(\theta_1, \varphi_1)$ and the $\phi^{(2)}_{j_2 m_2}(\theta_2, \varphi_2)$ are spherical harmonics.

In the Landé interpretation, the function $\Psi_{j_1 j_2 j M}(\theta_1, \varphi_1, \theta_2, \varphi_2)$ is a probability amplitude. Its expression in terms of an expansion must be of the general structure of Eq. (3). Therefore, we rewrite Eq. (6) in the following way:

$$\Psi(j_1, j_2, j, M; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1} \chi(j_1, j_2, j, M; j_1, m_1, j_2, m_2)$$

$$\times \Phi(j_1, m_1, j_2, m_2; \theta_1, \varphi_1, \theta_2, \varphi_2), \tag{7}$$

where

$$\chi(j_1, j_2, j, M; j_1, m_1, j_2, m_2) = C(j_1 j_2 j; m_1 m_2 M) \tag{8}$$

and

$$\Phi(j_1, m_1, j_2, m_2; \theta_1, \varphi_1, \theta_2, \varphi_2) = \phi^{(1)}_{j_1 m_1}(\theta_1, \varphi_1) \phi^{(2)}_{j_2 m_2}(\theta_2, \varphi_2). \tag{9}$$

Then the various quantities have the following interpretations:

The function $\Psi(j_1, j_2, j, M; \theta_1, \varphi_1, \theta_2, \varphi_2)$ is a probability amplitude characterized by an initial state corresponding to the quantum numbers (j_1, j_2, j, M), and a final state corresponding to the eigenvalues $(\theta_1, \varphi_1, \theta_2, \varphi_2)$. In the initial state, j is the quantum number for the total angular momentum, $M\hbar$ is the projection of the total angular momentum along the z direction, j_1 is the quantum number

of subsystem 1 and j_2 is the quantum number of subsystem 2. In the state which results from the measurement, (θ_1, φ_1) is the angular position of subsystem 1, while (θ_2, φ_2) is the angular position of subsystem 2. Thus, this probability amplitude gives the probability for obtaining specified angular positions of systems 1 and 2 upon measurement if the initial state of the compound system is defined by the quantum numbers (j_1, j_2, j, M).

The Clebsch-Gordan coefficient $\chi(j_1, j_2, j, M; j_1, m_1, j_2, m_2)$ is a probability amplitude characterized by an initial state corresponding to (j_1, j_2, j, M) and a final state defined by (j_1, m_1, j_2, m_2). In the state resulting from the measurement, the angular momentum quantum number of subsystem 1 is j_1, while its component in the z direction is $m_1\hbar$, and the angular momentum quantum number of subsystem 2 is j_2, while its z projection is $m_2\hbar$. This probability amplitude thus gives the probability of obtaining specified projections of the angular momenta of the subsystems along the z axis starting from a state of the compound system defined by the quantum numbers (j_1, j_2, j, M).

The function $\Phi(j_1, m_1, j_2, m_2; \theta_1, \varphi_1, \theta_2, \varphi_2)$ is a probability amplitude with an initial state defined by (j_1, m_1, j_2, m_2) and a final state defined by the eigenvalues $(\theta_1, \varphi_1, \theta_2, \varphi_2)$. This probability amplitude thus gives the probability of obtaining specified angular positions of the subsystems starting from a state characterized by specified projections of these subsystems along the z direction.

In labelling the various probability amplitudes, we may reduce on the clutter by suppressing those quantum numbers which do not change at all during the measurement. Thus, we omit j_1 and j_2. However, we retain the subscript j because for given j_1 and j_2, several values of j are possible within the limits

$$j_1 + j_2 \leq j \leq |j_1 - j_2|. \tag{10}$$

With these changes, Eq. (7) becomes

$$\Psi(j, M; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1} \chi(j, M; m_1, m_2)\Phi(m_1, m_2; \theta_1, \varphi_1, \theta_2, \varphi_2). \tag{11}$$

We have elsewhere interpreted the probability amplitude $\Psi(j, M; \theta_1, \varphi_1, \theta_2, \varphi_2)$ as a special form of the probability amplitude $\Psi(j(\theta, \varphi), M; \theta_1, \varphi_1, \theta_2, \varphi_2)$[1]. The former quantity is specialized because it pertains to a situation where projections of the total angular momentum are measured with respect to the z direction (for which $\theta = \varphi = 0$), while the latter corresponds to these projections being measured with respect to the arbitrary vector $\widehat{\mathbf{a}}$ whose polar angles are (θ, φ). To define the generalized probability amplitude corresponding to the latter case, we add the superscript $\widehat{\mathbf{a}}$ to M . The expansion for the generalized probability amplitude is thus

$$\Psi(j, M^{\widehat{\mathbf{a}}}; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{m_1, m_2} \chi(j, M^{\widehat{\mathbf{a}}}; m_1, m_2)\Phi(m_1, m_2; \theta_1, \varphi_1, \theta_2, \varphi_2). \tag{12}$$

In this expansion, the projections of the angular momenta of the subsystems are not necessarily measured with respect to the direction $\widehat{\mathbf{a}}$. In fact the projection of subsystem 1 need not be measured with respect to the same vector as the projection of subsystem 2. In general, the projection of the angular momentum of subsystem 1 is measured relative to the direction $\widehat{\mathbf{g}}_1$ and that of subsystem 2 with respect to the direction $\widehat{\mathbf{g}}_1$. For this reason, the condition that $\chi(j, M^{\widehat{\mathbf{a}}}; m_1, m_2)$ vanishes unless $m_1 + m_2 = M$ is generally not satisfied. Furthermore, the summation is generally a double summation, running over indices corresponding to the spin projections of subsystems 1 and 2 respectively. If the projections of subsystem 1 with respect to

4

$\widehat{\mathbf{g}}_1$ are $(m_1)_\alpha^{\widehat{(\mathbf{g}_1)}}$, while those of subsystem 2 with respect to $\widehat{\mathbf{g}}_2$ are $(m_2)_{\alpha'}^{\widehat{(\mathbf{g}_2)}}$, Eq. (12) is modified to

$$\Psi(j, M^{\widehat{(\mathbf{a})}}; \theta_1, \varphi_1, \theta_2, \varphi_2) = \sum_{\alpha, \alpha'} \chi(j, M^{\widehat{(\mathbf{a})}}; (m_1)_\alpha^{\widehat{(\mathbf{g}_1)}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{g}_2)}})$$

$$\times \Phi((m_1)_\alpha^{\widehat{(\mathbf{g}_1)}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{g}_2)}}; \theta_1, \varphi_1, \theta_2, \varphi_2). \tag{13}$$

But in the special event that $\widehat{\mathbf{g}}_1 = \widehat{\mathbf{g}}_2 = \widehat{\mathbf{a}}$, then the functions $\chi(j, M^{\widehat{(\mathbf{a})}}; (m_1)_\alpha^{\widehat{(\mathbf{a})}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{a})}})$ are generalized Clebsch-Gordan coefficients[1] and it is once more true that

$$(m_1)_\alpha^{\widehat{(\mathbf{a})}} + (m_2)_{\alpha'}^{\widehat{(\mathbf{a})}}) = M^{\widehat{(\mathbf{a})}}. \tag{14}$$

3.2 Theory for Spin

We now consider the case of spin. For a measurement on a simple system, the initial state is defined by a spin projection with respect to a given initial direction, while the final state is defined by a spin projection with respect to a new final direction. Thus, in the theory outlined in the previous section, probability amplitudes corresponding to spin projection measurements replace the spherical harmonics.

For a system of compounded spin, the total spin is

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2. \tag{15}$$

The quantum numbers of the spins are s, s_1 and s_2 for the total system, subsystem 1 and subsystem 2, respectively.

Suppose that the projections of the combined spin are initially known with respect to the direction of the vector $\widehat{\mathbf{a}}$, whose polar angles are (θ, φ). Let the projection of the total spin in that direction be $M_i^{\widehat{(\mathbf{a})}}\hbar$. We proceed to measure the projection of the spin of subsystem 1 with respect to the direction $\widehat{\mathbf{c}}_1$ (defined by the angles (θ_1, φ_1)) and the projection of the spin of subsystem 2 with respect to the direction $\widehat{\mathbf{c}}_2$ (polar angles (θ_2, φ_2)). The projections that result from the measurement are identified by their corresponding quantum numbers and the vectors with respect to which they are measured. Thus, the probability amplitude for this measurement is $\Psi(s, M_i^{\widehat{(\mathbf{a})}}; (m_1)_u^{\widehat{(\mathbf{c}_1)}}, (m_2)_v^{\widehat{(\mathbf{c}_2)}})$, where $(u, v = 1, 2, ...)$. The generalized probability amplitude Eq. (13) is expressed as

$$\Psi(j, M_i^{\widehat{(\mathbf{a})}}; (m_1)_u^{\widehat{(\mathbf{c}_1)}}, (m_2)_v^{\widehat{(\mathbf{c}_2)}}) = \sum_{\alpha, \alpha'} \chi(s, M_i^{\widehat{(\mathbf{a})}}; (m_1)_\alpha^{\widehat{(\mathbf{g}_1)}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{g}_2)}})$$

$$\times \Phi((m_1)_\alpha^{\widehat{(\mathbf{g}_1)}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{g}_2)}}; (m_1)_u^{\widehat{(\mathbf{c}_1)}}, (m_2)_v^{\widehat{(\mathbf{c}_2)}}). \tag{16}$$

Since the direction vectors $\widehat{\mathbf{g}}_1$ and $\widehat{\mathbf{g}}_2$ are arbitrary, they may be chosen for best convenience. The obvious choice is $\widehat{\mathbf{g}}_1 = \widehat{\mathbf{g}}_2 = \widehat{\mathbf{k}}$. We observe that if this is the case, and in addition $\widehat{\mathbf{a}} = \widehat{\mathbf{k}}$, then the χ's become Clebsch-Gordan coefficients. Since the Clebsch-Gordan coefficients vanish unless $(m_1)^{\widehat{(\mathbf{k})}} + (m_2)^{\widehat{(\mathbf{k})}} = M^{\widehat{(\mathbf{k})}}$, the double summation effectively becomes a single summation. In fact, if $\widehat{\mathbf{g}}_1$ and $\widehat{\mathbf{g}}_2$ are arbitrary, then the probability amplitudes $\chi(s, M_i^{\widehat{(\mathbf{a})}}; (m_1)_\alpha^{\widehat{(\mathbf{g}_1)}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{g}_2)}})$ and $\Psi(s, M_i^{\widehat{(\mathbf{a})}}; (m_1)_u^{\widehat{(\mathbf{c}_1)}}, (m_2)_v^{\widehat{(\mathbf{c}_2)}})$ are essentially identical, since they differ only in the choice of arbitrary vectors along which the spin projections of subsystems 1 and 2 are measured. In practice, it is essential to make the choice $\widehat{\mathbf{g}}_1 = \widehat{\mathbf{g}}_2 = \widehat{\mathbf{k}}$, in

order to convert $\chi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_\alpha^{(\widehat{\mathbf{g}_1})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{g}_2})})$ to $\chi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_\alpha^{(\widehat{\mathbf{k}})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{k}})})$ which can be expressed in terms of Clebsch-Gordan coefficients [1]. By this means it is possible to find an expression for $\Psi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})})$ (or equivalently $\chi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_\alpha^{(\widehat{\mathbf{g}_1})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{g}_2})}))$.

The actual form of the generalized spin probability amplitudes has been obtained in Ref. [1]. It is

$$\Psi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})}) = \tag{17}$$
$$\sum_{\alpha,\alpha'} \sum_l \zeta(s, M_i^{(\widehat{\mathbf{a}})}; s, M_l^{(\widehat{\mathbf{k}})}) \vartheta(s, M_l^{(\widehat{\mathbf{k}})}; (m_1)_\alpha^{(\widehat{\mathbf{k}})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{k}})})$$
$$\times \Phi((m_1)_\alpha^{(\widehat{\mathbf{k}})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{k}})}; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})}).$$

The quantities in expression (17) are defined as follows. The quantity $\zeta(s, M_i^{(\widehat{\mathbf{a}})}; s, M_l^{(\widehat{\mathbf{k}})})$ is the probability amplitude that if the total spin is s and its projection along the vector $\widehat{\mathbf{a}}$ is $M_i^{(\widehat{\mathbf{a}})}\hbar$, a measurement of its projection along the z axis gives $M_l^{(\widehat{\mathbf{k}})}\hbar$. The quantity $\vartheta(s, M_l^{(\widehat{\mathbf{k}})}; (m_1)_\alpha^{(\widehat{\mathbf{k}})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{k}})})$ is actually the standard Clebsch-Gordan coefficient for the case at hand. It is obtained from $\chi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_\alpha^{(\widehat{\mathbf{g}_1})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{g}_2})})$ by setting $\widehat{\mathbf{a}} = \widehat{\mathbf{g}}_1 = \widehat{\mathbf{g}}_2 = \widehat{\mathbf{k}}$.

For future convenience we rewrite Eq. (17) as

$$\Psi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})}) = \sum_{\alpha,\alpha'} \chi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_\alpha^{(\widehat{\mathbf{k}})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{k}})})$$
$$\times \Phi((m_1)_\alpha^{(\widehat{\mathbf{k}})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{k}})}; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})}), \tag{18}$$

where

$$\chi(s, M_i^{(\widehat{\mathbf{a}})}; (m_1)_\alpha^{(\widehat{\mathbf{k}})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{k}})}) = \sum_l \zeta(s, M_i^{(\widehat{\mathbf{a}})}; s, M_l^{(\widehat{\mathbf{k}})})$$
$$\times \vartheta(s, M_l^{(\widehat{\mathbf{k}})}; (m_1)_\alpha^{(\widehat{\mathbf{k}})}, (m_2)_{\alpha'}^{(\widehat{\mathbf{k}})}). \tag{19}$$

If we compare Eq. (18) with the fundamental expansion Eq. (3), we see that the intermediate observable which we are using to achieve the expansion is the combination of spin projections of systems 1 and 2 with respect to the z axis. The notation is simpler if we use the symbol B for this observable. However, because of the double summation, the symbol has two subscripts. If the subsystems 1 and 2 are both spin-1/2 systems, the values of B are

$$B_{11} = ((m_1)_1^{(\widehat{\mathbf{k}})}, (m_2)_1^{(\mathbf{k})}) = ((+\tfrac{1}{2})^{(\widehat{\mathbf{k}})}, (+\tfrac{1}{2})^{(\mathbf{k})}), \tag{20}$$

$$B_{12} = ((m_1)_1^{(\widehat{\mathbf{k}})}, (m_2)_2^{(\mathbf{k})}) = ((+\tfrac{1}{2})^{(\widehat{\mathbf{k}})}, (-\tfrac{1}{2})^{(\mathbf{k})}), \tag{21}$$

$$B_{21} = ((m_1)_2^{(\widehat{\mathbf{k}})}, (m_2)_1^{(\mathbf{k})}) = ((-\tfrac{1}{2})^{(\widehat{\mathbf{k}})}, (+\tfrac{1}{2})^{(\mathbf{k})}) \tag{22}$$

and

$$B_{22} = ((m_1)_2^{(\widehat{\mathbf{k}})}, (m_2)_2^{(\mathbf{k})}) = ((-\tfrac{1}{2})^{(\widehat{\mathbf{k}})}, (-\tfrac{1}{2})^{(\mathbf{k})}). \tag{23}$$

This makes it easier to denote values pertaining to just one subsystem. For a particular value $B_{\alpha\alpha'}$, we shall denote the value corresponding to the subsystem w by $(B_{\alpha\alpha'})_w$, where $w = 1, 2$. This means that for subsystem 1,

$$(B_{11})_1 = (B_{12})_1 = (+\frac{1}{2})^{\widehat{(\mathbf{k})}} \tag{24}$$

and

$$(B_{21})_1 = (B_{22})_1 = (-\frac{1}{2})^{\widehat{(\mathbf{k})}}, \tag{25}$$

while for subsystem 2,

$$(B_{11})_2 = (B_{21})_2 = (+\frac{1}{2})^{\widehat{(\mathbf{k})}} \tag{26}$$

and

$$(B_{12})_2 = (B_{22})_2 = (-\frac{1}{2})^{\widehat{(\mathbf{k})}}. \tag{27}$$

For the sake of convenience, we set $A_i = (s, M_i^{\widehat{(\mathbf{a})}})$. Then the probability amplitude Eq. (18) becomes

$$\Psi(A_i; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}) = \sum_{\alpha,\alpha'} \chi(A_i; B_{\alpha\alpha'})\Phi(B_{\alpha\alpha'}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}). \tag{28}$$

We remind ourselves that according to Eq. (9),

$$\Phi(B_{\alpha\alpha'}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}) = \Phi((m_1)_\alpha^{\widehat{(\mathbf{k})}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{k})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}})$$

$$= \phi_1((m_1)_\alpha^{\widehat{(\mathbf{k})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}})\phi_2((m_2)_{\alpha'}^{(\mathbf{k})} : (m_2)_v^{\widehat{(\mathbf{c_2})}}). \tag{29}$$

We remark that, purely for convenience, we have altered the notation slightly. Thus, $\phi_i = \phi^{(i)}$ $(i = 1, 2)$.

4   From Probability-Amplitude Mechanics to Matrix Mechanics

4.1   Theory For Simple Systems

In order to move from wave to matrix mechanics, an expansion of the eigenfunction or wave function in terms of some basis set is necessary. To obtain the matrix theory of orbital angular momentum we use the spherical harmonics as the basis set. This kind of procedure was thought impossible for spin, because spin is not describable by eigenfunctions resulting from an eigenvalue equation. But in our work[1, 7-11], we have shown how, by using the probability amplitudes for measurements on spin systems, we can derive the matrix treatment of spin in the same way as the matrix treatment of orbital angular momentum is obtained.

The relation Eq. (3) is the basis of the transformation from amplitude to matrix mechanics. We now review how we use it to obtain the matrix treatment of a simple quantum system. This review is needed because it is the foundation of the more involved derivation of the standard matrix treatment of compounded spin from generalized probability amplitudes.

Let us suppose that have a quantum system possessing the observables A, B and C. We assume that as we are measuring values of C, we are measuring values of a quantity $T(C)$ which is a function of C. Let $T(C)$ take upon measurement the values T_n determined by the values C_n of C. Thus $T_n = T(C_n)$. The expectation value of T is

$$\langle T(C) \rangle = \sum_{n=1}^{N} |\psi(A_i; C_n)|^2 T_n. \tag{30}$$

where N is the total number of eigenvalues of C.

If we use the expansions

$$\psi^*(A_i; C_n) = \sum_{j=1}^{N} \eta^*(A_i; B_j)\xi^*(B_j; C_n) \tag{31}$$

and

$$\psi(A_i; C_n) = \sum_{j'=1}^{N} \eta(A_i; B_{j'})\xi(B_{j'}; C_n), \tag{32}$$

we find

$$\langle T(C) \rangle = \sum_{j=1}^{N} \sum_{j'=1}^{N} \eta^*(A_i; B_j)T_{jj'}\eta(A_i; B_{j'}), \tag{33}$$

where

$$T_{jj'} = \sum_{n=1}^{N} \xi^*(B_j; C_n)T_n\xi(B_{j'}; C_n). \tag{34}$$

Hence

$$\langle T(C) \rangle = [\eta(A_i)]^{\dagger}[T][\eta(A_i)], \tag{35}$$

where the state is

$$[\eta(A_i)] = \begin{pmatrix} \eta(A_i; B_1) \\ \eta(A_i; B_2) \\ .. \\ \eta(A_i; B_N) \end{pmatrix}, \tag{36}$$

and the operator is

$$[T] = \begin{pmatrix} T_{11} & T_{12} & ... & T_{1N} \\ T_{21} & T_{22} & ... & T_{2N} \\ ... & ... & ... & ... \\ T_{N1} & T_{N2} & ... & T_{NN} \end{pmatrix}. \tag{37}$$

Thus, by means of the probability amplitudes for a quantum system, we can derive its matrix treatment. We note the convention of enclosing a quantity in brackets in order to denote its matrix representation.

4.2 Theory for Systems of Compounded Spin

The theory in the previous section will now be extended so as to yield the derivation of the matrix treatment of systems of compounded spin. As usual, we go through the expectation value in order to obtain the matrix form of the probability amplitudes for compounded spin, and of the operators for quantities that may be measured on the systems.

Suppose that a compounded spin is obtained by adding the spins s_1 and s_2. Suppose that initially the total spin is s and its projection with respect to the vector $\widehat{\mathbf{a}}$ is $M\hbar$. Subsequently, the spin projection of the spin of system 1 is measured along the vector $\widehat{\mathbf{c}}_1$ and the spin projection of the spin of system 2 is measured along the vector $\widehat{\mathbf{c}}_2$. At the same time, the quantity $R((m_1)^{(\widehat{\mathbf{c}}_1)}, (m_2)^{(\widehat{\mathbf{c}}_2)})$ is measured. This quantity is measured on the separate systems 1 and 2. It is constructed

from the quantity $r^{(1)}((m_1)^{(\widehat{\mathbf{c}_1})})$, which is measured on system 1 and the quantity $r^{(2)}((m_2)^{(\widehat{\mathbf{c}_2})})$ measured on system 2. Therefore, we write

$$R = R(r^{(1)}((m_1)^{(\widehat{\mathbf{c}_1})}), r^{(2)}((m_2)^{(\widehat{\mathbf{c}_2})}). \tag{38}$$

The values of $r^{(1)}((m_1)^{(\widehat{\mathbf{c}_1})})$ are independent of the values of $r^{(2)}((m_2)^{(\widehat{\mathbf{c}_2})})$: any value $r^{(1)}((m_1)_u^{(\widehat{\mathbf{c}_1})})$ and any value $r^{(2)}((m_2)_v^{(\widehat{\mathbf{c}_2})})$ can result together from the measurements. Therefore, the expectation value of R is

$$\langle R \rangle = \sum_u \sum_v \Psi^*(A_i; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})}) \Psi(A_i; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})})$$
$$\times R(r^{(1)}((m_1)_u^{(\widehat{\mathbf{c}_1})}), r^{(2)}((m_2)_v^{(\widehat{\mathbf{c}_2})})), \tag{39}$$

where $R(r^{(1)}((m_1)_u^{(\widehat{\mathbf{c}_1})}), r^{(2)}((m_2)_v^{(\widehat{\mathbf{c}_2})})$ is an actual value of $R((m_1)^{(\widehat{\mathbf{c}_1})}, (m_2)^{(\widehat{\mathbf{c}_2})})$.

The probability amplitude is given by Eq.(28). Using Eq. (29), we obtain the expansions

$$\Psi^*(A_i; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})}) = \sum_{\alpha,\alpha'} \chi^*(A_i; B_{\alpha\alpha'})$$
$$\times \phi_1^*((B_{\alpha\alpha'})_1; (m_1)_u^{(\widehat{\mathbf{c}_1})}) \phi_2^*((B_{\alpha\alpha'})_2; (m_2)_v^{(\widehat{\mathbf{c}_2})}) \tag{40}$$

and

$$\Psi(A_i; (m_1)_u^{(\widehat{\mathbf{c}_1})}, (m_2)_v^{(\widehat{\mathbf{c}_2})}) = \sum_{\beta,\beta'} \chi(A_i; B_{\beta\beta'})$$
$$\times \phi_1((B_{\beta\beta'})_1; (m_1)_u^{(\widehat{\mathbf{c}_1})}) \phi_2((B_{\beta\beta'})_2; (m_2)_v^{(\widehat{\mathbf{c}_2})}). \tag{41}$$

The expectation value becomes

$$\langle R \rangle = \sum_{\alpha,\alpha'} \sum_{\beta,\beta'} \chi^*(A_i; B_{\alpha\alpha'}) \chi(A_i; B_{\beta\beta'})$$
$$\times \sum_u \sum_v \{\phi_1^*((B_{\alpha\alpha'})_1; (m_1)_u^{(\widehat{\mathbf{c}_1})}) \phi_1((B_{\beta\beta'})_1; (m_1)_u^{(\widehat{\mathbf{c}_1})})$$
$$\times \phi_2^*((B_{\alpha\alpha'})_2; (m_2)_v^{(\widehat{\mathbf{c}_2})}) \phi_2((B_{\beta\beta'})_2; (m_2)_v^{(\widehat{\mathbf{c}_2})})$$
$$\times R(r^{(1)}((m_1)_u^{(\widehat{\mathbf{c}_1})}) r^{(2)}((m_2)_v^{(\widehat{\mathbf{c}_2})}))\}. \tag{42}$$

When R is factorizable, so that

$$R(r^{(1)}((m_1)^{(\widehat{\mathbf{c}_1})}), r^{(2)}((m_2)^{(\widehat{\mathbf{c}_2})}) = r^{(1)}((m_1)^{(\widehat{\mathbf{c}_1})}) r^{(2)}((m_2)^{(\widehat{\mathbf{c}_2})}), \tag{43}$$

we can write Eq. (42) as

$$\langle R \rangle = \sum_\alpha \sum_{\alpha'} \chi^*(A_i; B_{\alpha\alpha'}) \chi(A_i; B_{\beta\beta'}) \sum_u \phi_1^*((B_{\alpha\alpha'})_1; (m_1)_u^{(\widehat{\mathbf{c}_1})})$$
$$\times \phi_1((B_{\beta\beta'})_1; (m_1)_u^{(\widehat{\mathbf{c}_1})}) r^{(1)}((m_1)_u^{(\widehat{\mathbf{c}_1})})$$
$$\times \sum_v \phi_2^*((B_{\alpha\alpha'})_2; (m_2)_v^{(\widehat{\mathbf{c}_2})}) \phi_2((B_{\beta\beta'})_2; (m_2)_v^{(\widehat{\mathbf{c}_2})}) r^{(2)}((m_2)_v^{(\widehat{\mathbf{c}_2})}). \tag{44}$$

Thus

$$\langle R \rangle = \sum_{\alpha,\alpha'} \sum_{\beta,\beta'} \chi^*(A_i; B_{\alpha\alpha'}) I^{(1)}_{\alpha\alpha'\beta\beta'} I^{(2)}_{\alpha\alpha'\beta\beta'} \chi(A_i; B_{\beta\beta'}), \tag{45}$$

where

$$
\begin{aligned}
I^{(1)}_{\alpha\alpha'\beta\beta'} &= \sum_u \phi_1^*((B_{\alpha\alpha'})_1; (m_1)_u^{\widehat{(\mathbf{c_1})}}) r^{(1)}((m_1)_u^{\widehat{(\mathbf{c_1})}}) \\
&\times \phi_1((B_{\beta\beta'})_1; (m_1)_u^{\widehat{(\mathbf{c_1})}})
\end{aligned}
\tag{46}
$$

and

$$
\begin{aligned}
I^{(2)}_{\alpha\alpha'\beta\beta'} &= \sum_v \phi_2^*((B_{\alpha\alpha'})_2; (m_2)_v^{\widehat{(\mathbf{c_2})}}) r^{(2)}((m_2)_v^{\widehat{(\mathbf{c_2})}}) \\
&\times \phi_2((B_{\beta\beta'})_2; (m_2)_v^{\widehat{(\mathbf{c_2})}}).
\end{aligned}
\tag{47}
$$

In order to treat $I^{(1)}_{\alpha\alpha'\beta\beta'}$, we introduce the observable D which corresponds to spin projections of subsystem 1 with respect to the direction $\widehat{\mathbf{d}}$, whose polar angles are (θ_d, φ_d). We use this observable to expand ϕ_1 by means of formula (3). We note that the values D_p are

$$D_p = (m_1)_p^{\widehat{(\mathbf{d})}} \hbar. \tag{48}$$

The expansions of ϕ_1 and ϕ_1^* are

$$\phi_1^*((B_{\alpha\alpha'})_1; (m_1)_u^{\widehat{(\mathbf{c_1})}}) = \sum_p \eta_1^*((B_{\alpha\alpha'})_1; D_p) \xi_1^*(D_p; (m_1)_u^{\widehat{(\mathbf{c_1})}}) \tag{49}$$

and

$$\phi_1((B_{\beta\beta'})_1; (m_1)_u^{\widehat{(\mathbf{c_1})}}) = \sum_{p'} \eta_1((B_{\beta\beta'})_1; D_{p'}) \xi_1(D_{p'}; (m_1)_u^{\widehat{(\mathbf{c_1})}}). \tag{50}$$

Applying the theory outlined in Section 4.1, we obtain

$$I^{(1)}_{\alpha\alpha'\beta\beta'} = \sum_p \sum_{p'} \eta_1^*((B_{\alpha\alpha'})_1; D_p) r^{(1)}_{pp'} \eta_1((B_{\beta\beta'})_1; D_{p'}), \tag{51}$$

where

$$r^{(1)}_{pp'} = \sum_u \xi_1^*(D_p; (m_1)_u^{\widehat{(\mathbf{c_1})}}) r^{(1)}((m_1)_u^{\widehat{(\mathbf{c_1})}}) \xi_1(D_{p'}; (m_1)_u^{\widehat{(\mathbf{c_1})}}). \tag{52}$$

Hence

$$I^{(1)}_{\alpha\alpha'\beta\beta'} = [\eta_1((B_{\alpha\alpha'})_1)]^\dagger [r^{(1)}] [\eta_1((B_{\beta\beta'})_1)], \tag{53}$$

where

$$[\eta_1(B_{\alpha\alpha'})] = \begin{pmatrix} \eta_1((B_{\alpha\alpha'})_1; D_1) \\ \eta_1((B_{\alpha\alpha'})_1; D_2) \\ \vdots \\ \eta_1((B_{\alpha\alpha'})_1; D_N) \end{pmatrix} \tag{54}$$

and

$$[r^{(1)}] = \begin{pmatrix} r^{(1)}_{11} & r^{(1)}_{12} & .. & r^{(1)}_{1N} \\ r^{(1)}_{21} & r^{(1)}_{22} & .. & r^{(1)}_{2N} \\ .. & .. & .. & .. \\ r^{(1)}_{N1} & r^{(1)}_{N2} & .. & r^{(1)}_{NN} \end{pmatrix}. \tag{55}$$

Here N is the total number of states of the observable D. In the case where subsystem 1 is a spin-1/2 system, $N = 2$.

To deal with $I^{(2)}_{\alpha\alpha'\beta\beta'}$, we introduce the observable F defined by the unit vector $\widehat{\mathbf{f}}$ whose polar angles are (θ_f, φ_f). The eigenvalues of F are

$$F_q = (m_2)_q^{(\widehat{\mathbf{f}})}\hbar. \tag{56}$$

Expanding ϕ_2 over the states of F, we get

$$\phi_2^*(B_{\alpha\alpha'}; (m_2)_v^{(\widehat{\mathbf{c}_2})}) = \sum_q \eta_2^*((B_{\alpha\alpha'})_2; F_q)\xi_2^*(F_q; (m_2)_v^{(\widehat{\mathbf{c}_2})}) \tag{57}$$

and

$$\phi_2(B_{\beta\beta'}; (m_2)_v^{(\widehat{\mathbf{c}_2})}) = \sum_{q'} \eta_2((B_{\beta\beta'})_2; F_{q'})\xi_2(F_{q'}; (m_2)_v^{(\widehat{\mathbf{c}_2})}). \tag{58}$$

This means that

$$I^{(2)}_{\alpha\alpha'\beta\beta'} = \sum_q \sum_{q'} \eta_2^*((B_{\alpha\alpha'})_2; F_q)r^{(2)}_{qq'}\eta_2((B_{\beta\beta'})_2; F_{q'}), \tag{59}$$

where

$$r^{(2)}_{qq'} = \sum_v \xi_2^*(F_q; (m_2)_v^{(\widehat{\mathbf{c}_2})})r^{(2)}((m_2)_v^{(\widehat{\mathbf{c}_2})})\xi_2(F_{q'}; (m_2)_v^{(\widehat{\mathbf{c}_2})}). \tag{60}$$

In view of Eq. (59), the expression for $I^{(2)}_{\alpha\alpha'\beta\beta'}$ is

$$I^{(2)}_{\alpha\alpha'\beta\beta'} = [\eta_2((B_{\alpha\alpha'})_2)]^\dagger[r^{(2)}][\eta_2((B_{\beta\beta'})_2)], \tag{61}$$

where

$$[\eta_2((B_{\alpha\alpha'})_2)] = \begin{pmatrix} \eta_2((B_{\alpha\alpha'})_2; F_1) \\ \eta_2((B_{\alpha\alpha'})_2; F_2) \\ \vdots \\ \eta_2((B_{\alpha\alpha'})_2; F_M) \end{pmatrix} \tag{62}$$

and

$$[r^{(2)}] = \begin{pmatrix} r^{(2)}_{11} & r^{(2)}_{12} & .. & r^{(2)}_{1M} \\ r^{(2)}_{21} & r^{(2)}_{22} & .. & r^{(2)}_{2M} \\ .. & .. & .. & .. \\ r^{(2)}_{M1} & r^{(2)}_{M2} & .. & r^{(2)}_{MM} \end{pmatrix}. \tag{63}$$

Here M is the number of states of F.

Collecting all the results together, we find that

$$\begin{aligned}
\langle R \rangle &= \sum_{\alpha,\alpha'} \sum_{\beta,\beta'} \chi^*(A_i; B_{\alpha\alpha'})[\eta_1((B_{\alpha\alpha'})_1)]^\dagger[r^{(1)}][\eta_1((B_{\beta\beta'})_1)] \\
&\quad \times [\eta_2((B_{\alpha\alpha'})_2)]^\dagger[r^{(2)}][\eta_2((B_{\beta\beta'})_2)]\chi(A_i; B_{\beta\beta'}) \\
&= \left(\sum_{\alpha,\alpha'} \chi^*(A_i; B_{\alpha\alpha'})[\eta_1((B_{\alpha\alpha'})_1)]^\dagger[\eta_2((B_{\alpha\alpha'})_2)]^\dagger \right)[r^{(1)}][r^{(2)}] \\
&\quad \times \left(\sum_{\beta,\beta'} \chi(A_i; B_{\beta\beta'})[\eta_1((B_{\beta\beta'})_1)][\eta_2((B_{\beta\beta'})_2)] \right) \\
&= [\Psi(A_i; (m_1)^{(\widehat{\mathbf{c}_1})}, (m_2)^{(\widehat{\mathbf{c}_2})})]^\dagger[R(r^{(1)}((m_1)^{(\widehat{\mathbf{c}_1})}), r^{(2)}((m_2)^{(\widehat{\mathbf{c}_2})}))] \\
&\quad \times [\Psi(A_i; (m_1)^{(\widehat{\mathbf{c}_1})}, (m_2)^{(\widehat{\mathbf{c}_2})})],
\end{aligned} \tag{64}$$

11

where

$$[\Psi(A_i; (m_1)^{\widehat{(\mathbf{c}_1)}}, (m_2)^{\widehat{(\mathbf{c}_2)}})] = \left(\sum_{\alpha,\alpha'} \chi(A_i; B_{\alpha\alpha'})[\eta_1((B_{\alpha\alpha'})_1)][\eta_2((B_{\alpha\alpha'})_2)] \right) \quad (65)$$

and

$$[R(r^{(1)}((m_1)^{\widehat{\mathbf{c}_1}}), r^{(2)}((m_2)^{\widehat{(\mathbf{c}_2)}}))] = [r^{(1)}][r^{(2)}]. \quad (66)$$

We see that in Eq. (65) we have obtained the generalized vector state corresponding to the probability amplitude $\Psi(A_i; (m_1)_u^{\widehat{(\mathbf{c}_1)}}, (m_2)_v^{\widehat{(\mathbf{c}_2)}})$. In addition, in Eqs. (52), 55, (60) and (63), we have obtained the generalized operator for any observable of the system which is a function of the spin projections of the subsystems.

We note that by definition, the vectors relating to subsystem 1 act only on one another and on the operator corresponding to this subsystem. The same holds for the quantities corresponding to subsystem 2. To emphasize this fact, we introduce the labels 1 and 2 to distinguish the corresponding quantities. Thus, we get

$$[\Psi(A_i; (m_1)^{\widehat{(\mathbf{c}_1)}}, (m_2)^{\widehat{(\mathbf{c}_2)}})] = \left(\sum_{\alpha,\alpha'} \chi(A_i; B_{\alpha\alpha'})[\eta_1((B_{\alpha\alpha'})_1)]_1[\eta_2((B_{\alpha\alpha'})_2)]_2 \right) \quad (67)$$

and

$$[R(r^{(1)}((m_1)^{\widehat{(\mathbf{c}_1)}}), r^{(2)}((m_2)^{\widehat{(\mathbf{c}_2)}}))] = [r^{(1)}((m_1)^{\widehat{(\mathbf{c}_1)}})]_1[r^{(2)}((m_2)^{\widehat{(\mathbf{c}_2)}})]_2. \quad (68)$$

These results are dependent on the condition that R is factorizable; if this is not the case, it is not so easy to transform Eq. (42) to matrix form. But in that case, we can use the alternative approach of Ref. [1]; we then end up with 3- or 4-dimensional matrix representations which appply whether R is factorizable or not.

5 Application to Actual Systems

5.1 The Matrix Operator

The results derived in the last section will now be used to obtain specific operators and vectors. For a particular case, the operator is calculated by explicitly working out the matrix elements $r_{pp'}^{(1)}$ and $r_{qq'}^{(2)}$. The cases at hand are of a system of total spin 0, and of a system of total spin 1 (with three possible values of the magnetic quantum number) obtained by adding two spins of 1/2 each. The form of the operator is independent of whether the total spin is 0 or 1. We therefore derive this quantity first.

We start with the matrix element $r_{pp'}^{(1)}$. In order to obtain this quantity, we require the forms of the probability amplitudes ξ. Since both systems 1 and 2 are spin-1/2 systems, these are obtained from the generalized spin-1/2 amplitudes, whose explicit forms we have already worked out [7,8,10].

We first recall the details of the probability amplitudes for spin 1/2. We consider system 1. Let the spin projection be initially known with respect to $\mathbf{d}$; it is subsequently measured with respect to $\widehat{\mathbf{c}}_1$. The probability amplitude that it will be found upon measurement to be up with respect to $\widehat{\mathbf{c}}_1$ is $\xi_1((+\frac{1}{2})^{\widehat{(\mathbf{d})}}; (+\frac{1}{2})^{\widehat{(\mathbf{c}_1)}})$. The other three probability amplitudes are therefore $\xi_1((+\frac{1}{2})^{\widehat{(\mathbf{d})}}; (-\frac{1}{2})^{\widehat{(\mathbf{c}_1)}})$, $\xi_1((-\frac{1}{2})^{\widehat{(\mathbf{d})}}; (+\frac{1}{2})^{\widehat{(\mathbf{c}_1)}})$ and $\xi_1((-\frac{1}{2})^{\widehat{(\mathbf{d})}}; (-\frac{1}{2})^{\widehat{(\mathbf{c}_1)}})$. These probability amplitudes come in a variety of forms,

depending on the phase choice made when they are being derived [10]. One form is the following:

$$\xi_1((+\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) = \cos\theta_d/2\cos\theta_1/2 + e^{i(\varphi_d-\varphi_1)}\sin\theta_d/2\sin\theta_1/2, \qquad (69)$$

$$\xi_1((+\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) = -\cos\theta_d/2\sin\theta_1/2 + e^{i(\varphi_d-\varphi_1)}\sin\theta_d/2\cos\theta_1/2, \qquad (70)$$

$$\xi_1((-\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) = -\sin\theta_d/2\cos\theta_1/2 + e^{i(\varphi_d-\varphi_1)}\cos\theta_d/2\sin\theta_1/2 \qquad (71)$$

and

$$\xi_1((-\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) = \sin\theta_d/2\sin\theta_1/2 + e^{i(\varphi_d-\varphi_1)}\cos\theta_d/2\cos\theta_1/2. \qquad (72)$$

Since in this case system 2 is also a spin-1/2 system, the probability amplitudes corresponding to it are identical in form to Eqs. (69) - (72). To obtain them, we merely make the following change to the labels: 2 replaces subscript 1; $\widehat{\mathbf{f}}$ replaces $\widehat{\mathbf{d}}$, so that f replaces d; and $\widehat{\mathbf{c}}_2$ replaces $\widehat{\mathbf{c}}_1$.

For the case of spin 1/2, the summation over u which appears in the expression for $r^{(1)}_{pp'}$ contains only two terms. $u=1$ corresponds to the outcome $(+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}$ while $u=2$ corresponds to $(-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}$. Thus

$$
\begin{aligned}
r^{(1)}_{11} &= \left|\xi_1((+\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})})\right|^2 r^{(1)}((+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) \\
&\quad + \left|\xi_1((+\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})})\right|^2 r^{(1)}((-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}). \qquad (73)
\end{aligned}
$$

The values of the summation indices p and $p^{'}$ are such that $p, p^{'} = +1$ corresponds to $(+\tfrac{1}{2})^{(\widehat{\mathbf{d}})}$, while $p, p^{'} = 2$ corresponds to $(-\tfrac{1}{2})^{(\widehat{\mathbf{d}})}$. Hence

$$
\begin{aligned}
r^{(1)}_{11} &= [\cos^2(\theta_d-\theta_1)/2 - \sin\theta_d\sin\theta_1\sin^2(\varphi_d-\varphi_1)/2] r^{(1)}((+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) \\
&\quad + [\sin^2(\theta_d-\theta_1)/2 + \sin\theta_d\sin\theta_1\sin^2(\varphi_d-\varphi_1)/2] r^{(1)}((-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}). \qquad (74)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
r^{(1)}_{12} &= \xi_1^*((+\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})})\xi_1((-\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) r^{(1)}((+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) \\
&\quad + \xi_1^*((+\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})})\xi_1((-\tfrac{1}{2})^{(\widehat{\mathbf{d}})};(-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) r^{(1)}((-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) \\
&= [-\tfrac{1}{2}\sin\theta_d\cos\theta_1 + \tfrac{1}{2}\sin\theta_1\cos\theta_d\cos(\varphi_d-\varphi_1) \\
&\quad + \tfrac{i}{2}\sin\theta_1\sin(\varphi_d-\varphi_1)] r^{(1)}((+\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}) \\
&\quad + [\tfrac{1}{2}\sin\theta_d\cos\theta_1 - \tfrac{1}{2}\sin\theta_1\cos\theta_d\cos(\varphi_d-\varphi_1) \\
&\quad - \tfrac{i}{2}\sin\theta_1\sin(\varphi_d-\varphi_1)] r^{(1)}((-\tfrac{1}{2})^{(\widehat{\mathbf{c}_1})}), \qquad (75)
\end{aligned}
$$

13

$$r_{21}^{(1)} = r_{12}^{(1)*} \tag{76}$$

and

$$r_{22}^{(1)} = [\sin^2(\theta_d - \theta_1)/2 + \sin\theta_d \sin\theta_1 \sin^2(\varphi_d - \varphi_1)/2]r^{(1)}((+\tfrac{1}{2})^{\widehat{\mathbf{c_1}}})$$

$$+[\cos^2(\theta_d - \theta_1)/2 - \sin\theta_d \sin\theta_1 \sin^2(\varphi_d - \varphi_1)/2]r^{(1)}((-\tfrac{1}{2})^{\widehat{\mathbf{c_1}}}). \tag{77}$$

The elements of $[r^{(2)}]_2$ are identical in form to those of $[r^{(1)}]_1$. The difference is that in system 2 the vector $\widehat{\mathbf{f}}$ plays the role that the vector $\widehat{\mathbf{d}}$ plays in system 1. Thus, wherever d appears, it is replaced by f. Wherever the label 1 appears, it is replaced by 2. Wherever $(\pm\tfrac{1}{2})^{\widehat{\mathbf{c_1}}}$ appears, it is replaced by $(\pm\tfrac{1}{2})^{\widehat{\mathbf{c_2}}}$. Finally wherever $r^{(1)}$ appears, it is replaced by $r^{(2)}$. Thus, the elements of $[r^{(2)}]_2$ are

$$r_{11}^{(2)} = [\cos^2(\theta_f - \theta_2)/2 - \sin\theta_d \sin\theta_1 \sin^2(\varphi_f - \varphi_2)/2]r^{(2)}((+\tfrac{1}{2})^{\widehat{\mathbf{c_2}}})$$

$$+[\sin^2(\theta_f - \theta_2)/2 + \sin\theta_d \sin\theta_1 \sin^2(\varphi_f - \varphi_2)/2]r^{(2)}((-\tfrac{1}{2})^{\widehat{\mathbf{c_2}}}), \tag{78}$$

$$r_{12}^{(2)} = [-\frac{1}{2}\sin\theta_f \cos\theta_2 + \frac{1}{2}\sin\theta_2 \cos\theta_f \cos(\varphi_f - \varphi_2)$$

$$+\frac{i}{2}\sin\theta_2 \sin(\varphi_f - \varphi_2)]r^{(2)}((+\tfrac{1}{2})^{\widehat{\mathbf{c_2}}})$$

$$+[\frac{1}{2}\sin\theta_f \cos\theta_2 - \frac{1}{2}\sin\theta_2 \cos\theta_f \cos(\varphi_f - \varphi_2)$$

$$-\frac{i}{2}\sin\theta_2 \sin(\varphi_f - \varphi_2)]r^{(2)}((-\tfrac{1}{2})^{\widehat{\mathbf{c_2}}}), \tag{79}$$

$$r_{21}^{(2)} = r_{12}^{(2)*} \tag{80}$$

and

$$r_{22}^{(2)} = [\sin^2(\theta_f - \theta_2)/2 + \sin\theta_f \sin\theta_2 \sin^2(\varphi_f - \varphi_2)/2]r^{(2)}((+\tfrac{1}{2})^{\widehat{\mathbf{c_2}}})$$

$$+[\cos^2(\theta_f - \theta_2)/2 - \sin\theta_f \sin\theta_2 \sin^2(\varphi_f - \varphi_2)/2]r^{(2)}((-\tfrac{1}{2})^{\widehat{\mathbf{c_2}}}). \tag{81}$$

5.2 The Vector States

5.2.1 The Triplet State

We start our calculation of the states by looking at the probability amplitudes corresponding to the triplet state, defined by the quantum numbers $s = 1$, $M^{\widehat{(\mathbf{a})}} = 0, \pm 1$. We first consider the case $M^{\widehat{(\mathbf{a})}} = 1$.

The $M^{\widehat{(\mathbf{a})}} = 1$ State   For this case, we write,

$$\Psi(1, 1^{\widehat{(\mathbf{a})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}) = \Psi(s = 1, M = 1^{\widehat{(\mathbf{a})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}), \tag{82}$$

and the generalized probability amplitude is [1]

$$\Psi(1, 1^{\widehat{(\mathbf{a})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}) =$$

$$\sum_{\alpha,\alpha'} \left[\sum_l \zeta(1, 1^{\widehat{(\mathbf{a})}}; 1, M_l^{\widehat{(\mathbf{k})}}) \vartheta(1, M_l^{\widehat{(\mathbf{k})}}; (m_1)_\alpha^{\widehat{(\mathbf{k})}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{k})}}) \right]$$

$$\times \Phi((m_1)_\alpha^{\widehat{(\mathbf{k})}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{k})}}); (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}). \tag{83}$$

Thus,

$$\Psi(1, 1^{\widehat{(\mathbf{a})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}) = \sum_{\alpha,\alpha'} \chi(1, 1^{\widehat{(\mathbf{a})}}; B_{\alpha\alpha'})$$

$$\times \Phi(B_{\alpha\alpha'}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}), \tag{84}$$

where

$$\chi(1, 1^{\widehat{(\mathbf{a})}}; B_{\alpha\alpha'}) = \sum_l \zeta(1, 1^{\widehat{(\mathbf{a})}}; 1, M_l^{\widehat{(\mathbf{k})}})\vartheta(1, M_l^{\widehat{(\mathbf{k})}}; B_{\alpha\alpha'}). \tag{85}$$

With the probability amplitude expressed in the form Eq. (84), which is identical in form to Eq. (28), the transformation to matrix form is straightforwardly achieved. We find that the matrix form of the probability amplitude is

$$[\Psi(1, 1^{\widehat{(\mathbf{a})}}; (m_1)^{\widehat{(\mathbf{c_1})}}, (m_2)^{\widehat{(\mathbf{c_2})}})] = \sum_{\alpha,\alpha'} \chi(1, 1^{\widehat{(\mathbf{a})}}; B_{\alpha\alpha'})[\eta_1((B_{\alpha\alpha'})_1)]_1[\eta_2((B_{\alpha\alpha'})_2)]_2. \tag{86}$$

As $B_{\alpha\alpha'}$ takes the values Eqs. (20) - (23), we have

$$\begin{aligned}
[\Psi(1, 1^{\widehat{(\mathbf{a})}}; (m_1)^{\widehat{(\mathbf{c_1})}}, (m_2)^{\widehat{(\mathbf{c_2})}})] = \ & \chi(1, 1^{\widehat{(\mathbf{a})}}; (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) \\
& \times [\eta_1((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2 \\
& + \chi(1, 1^{\widehat{(\mathbf{a})}}; (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})[\eta_1((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2 \\
& + \chi(1, 1^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})[\eta_1((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2 \\
& + \chi(1, 1^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})[\eta_1((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2,
\end{aligned} \tag{87}$$

where

$$[\eta_1((\pm\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1 = \begin{pmatrix} \eta_1((\pm\tfrac{1}{2})^{\widehat{(\mathbf{k})}}; (+\tfrac{1}{2})^{\widehat{(\mathbf{d})}}) \\ \eta_1((\pm\tfrac{1}{2})^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{d})}}) \end{pmatrix}_1 \tag{88}$$

and

$$[\eta_2((\pm\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2 = \begin{pmatrix} \eta_2((\pm\tfrac{1}{2})^{\widehat{(\mathbf{k})}}); (+\tfrac{1}{2})^{\widehat{(\mathbf{f})}}) \\ \eta_2((\pm\tfrac{1}{2})^{\widehat{(\mathbf{k})}}); (-\tfrac{1}{2})^{\widehat{(\mathbf{f})}}) \end{pmatrix}_2. \tag{89}$$

The η_1's and η_2's are known. They are just the spin-1/2 probability amplitudes and are essentially identical to the ξ's, Eqs. (69) - (72). The only difference is in the direction vectors. In the labelling of the arguments for the η's, the initial direction corresponds to the z axis, so that its direction vector is $\widehat{\mathbf{k}}$; the final directions are defined by $\widehat{\mathbf{d}}$ and $\widehat{\mathbf{f}}$ for system 1 and system 2 respectively. From Eqs. (69) -(72), with the arguments appropriately changed, we get

$$[\eta_1((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1 = \begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1, \tag{90}$$

$$[\eta_1(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}]_1 = \begin{pmatrix} \sin\theta_d/2\, e^{-i\varphi_d} \\ \cos\theta_d/2\, e^{-i\varphi_d} \end{pmatrix}_1, \tag{91}$$

$$[\eta_2(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}]_2 = \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 \tag{92}$$

15

and

$$[\eta_2((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2 = \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2. \tag{93}$$

It only remains to compute the χ's. According to Eq. (85)

$$\chi(1,1^{\widehat{(\mathbf{a})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = \chi(1,1^{\widehat{(\mathbf{a})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$

$$= \zeta(1,1^{\widehat{(\mathbf{a})}};1,1^{\widehat{(\mathbf{k})}})\vartheta(1,1^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$

$$+\zeta(1,1^{\widehat{(\mathbf{a})}};1,0^{\widehat{(\mathbf{k})}})\vartheta(1,0^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$

$$+\zeta(1,1^{\widehat{(\mathbf{a})}};1,(-1)^{\widehat{(\mathbf{k})}})\vartheta(1,(-1)^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}). \tag{94}$$

The angles defining $\widehat{\mathbf{a}}$ are (θ,φ). As we have shown [1,9],

$$\zeta(1,1^{\widehat{(\mathbf{a})}};1,1^{\widehat{(\mathbf{k})}}) = \cos^2\theta/2 e^{-i\varphi} \tag{95}$$

$$\zeta(1,1^{\widehat{(\mathbf{a})}};1,0^{\widehat{(\mathbf{k})}}) = \frac{1}{\sqrt{2}}\sin\theta \tag{96}$$

and

$$\zeta(1,1^{\widehat{(\mathbf{a})}};1,(-1)^{\widehat{(\mathbf{k})}}) = \sin^2\theta/2 e^{i\varphi}. \tag{97}$$

The ϑ's are Clebsch-Gordan coefficients. As a result

$$\vartheta(1,1^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1;\tfrac{1}{2}\tfrac{1}{2}1) = 1, \tag{98}$$

$$\vartheta(1,0^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1;\tfrac{1}{2}\tfrac{1}{2}0) = 0 \tag{99}$$

and

$$\vartheta(1,(-1)^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1;\tfrac{1}{2}\tfrac{1}{2}-1) = 0. \tag{100}$$

Hence,

$$\chi(1,1^{\widehat{(\mathbf{a})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = \frac{1}{2}\sin\theta. \tag{101}$$

Similarly,

$$\chi(1,1^{\widehat{(\mathbf{a})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = \zeta(1,1^{\widehat{(\mathbf{a})}};1,1^{\widehat{(\mathbf{k})}})\vartheta(1,1^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$

$$+\zeta(1,1^{\widehat{(\mathbf{a})}};1,0^{\widehat{(\mathbf{k})}})\vartheta(1,0^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$

$$+\zeta(1,1^{\widehat{(\mathbf{a})}};1,(-1)^{\widehat{(\mathbf{k})}})\vartheta(1,(-1)^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}). \tag{102}$$

This is the same as the expression for $\chi(1,1^{\widehat{(\mathbf{a})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$, except for the change in the ϑ's. Since

$$\vartheta(1,+1^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1;\tfrac{1}{2},-\tfrac{1}{2}1) = 0, \tag{103}$$

$$\vartheta(1,(-1)^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1;\tfrac{1}{2},-\tfrac{1}{2},-1) = 0 \tag{104}$$

and

$$\vartheta(1,0^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1;\tfrac{1}{2},-\tfrac{1}{2}0) = \frac{1}{\sqrt{2}}, \tag{105}$$

we find that

$$\chi(1,1^{\widehat{(\mathbf{a})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = \frac{1}{2}\sin\theta. \tag{106}$$

16

In the same way

$$\chi(1, +1^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = \zeta(1, +1^{\widehat{(\mathbf{a})}}; 1, +1^{\widehat{(\mathbf{k})}})\vartheta(1, +1^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$
$$+\zeta(1, +1^{\widehat{(\mathbf{a})}}; 1, 0^{\widehat{(\mathbf{k})}})\vartheta(1, 0^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$
$$+\zeta(1, +1^{\widehat{(\mathbf{a})}}; 1, (-1)^{\widehat{(\mathbf{k})}})\vartheta(1, (-1)^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}). \tag{107}$$

In this case, we have

$$\vartheta(1, +1^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1; -\tfrac{1}{2}\tfrac{1}{2}1) = 0, \tag{108}$$

$$\vartheta(1, (-1)^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1; -\tfrac{1}{2}\tfrac{1}{2}, -1) = 0 \tag{109}$$

and

$$\vartheta(1, 0^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1; -\tfrac{1}{2}\tfrac{1}{2}0) = \frac{1}{\sqrt{2}}. \tag{110}$$

Thus,

$$\chi(1, +1^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = \frac{1}{2}\sin\theta. \tag{111}$$

Finally

$$\chi(1, 1^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) =$$
$$\zeta(1, 1^{\widehat{(\mathbf{a})}}; 1, 1^{\widehat{(\mathbf{k})}})\vartheta(1, +1^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$
$$+\zeta(1, 1^{\widehat{(\mathbf{a})}}; 1, 0^{\widehat{(\mathbf{k})}})\vartheta(1, 0^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})$$
$$+\zeta(1, 1^{\widehat{(\mathbf{a})}}; 1, (-1)^{\widehat{(\mathbf{k})}})\vartheta(1, (-1)^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}). \tag{112}$$

With

$$\vartheta(1, 1^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1; -\tfrac{1}{2}, -\tfrac{1}{2}1) = 0, \tag{113}$$

$$\vartheta(1, (-1)^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1; -\tfrac{1}{2}, -\tfrac{1}{2}, -1) = 1 \tag{114}$$

and

$$\vartheta(1, 0^{\widehat{(\mathbf{k})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}1; -\tfrac{1}{2}, -\tfrac{1}{2}0) = 0. \tag{115}$$

we obtain

$$\chi(1, 1^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = \sin^2\theta/2e^{i\varphi}. \tag{116}$$

Combining all these results together, we find that the matrix state for $s = 1$, $M^{\widehat{(\mathbf{a})}} = 1$ is

$$[\Psi(1, 1^{\widehat{(\mathbf{a})}}; (m_1)^{\widehat{\mathbf{c}_1}}, (m_2)^{\widehat{\mathbf{c}_2}})] = \cos^2\theta/2e^{-i\varphi}[\eta_1((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2$$
$$+\frac{1}{2}\sin\theta[\eta_1((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2$$
$$+\frac{1}{2}\sin\theta[\eta_1((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2$$
$$+\sin^2\theta/2e^{i\varphi}[\eta_1((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2. \tag{117}$$

Thus the generalised form of the triplet state for $M^{\widehat{(\mathbf{a})}} = 1$ is,

$$
\begin{aligned}
[\Psi(1,1^{\widehat{(\mathbf{a})}};(m_1)^{\widehat{(\mathbf{c}_1)}},(m_2)^{\widehat{(\mathbf{c}_2)}})] = \quad & \cos^2\theta/2 e^{-i\varphi} \begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1 \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 \\
& +\frac{1}{2}\sin\theta \begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1 \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2 \\
& +\frac{1}{2}\sin\theta \begin{pmatrix} \sin\theta_d/2 e^{-i\varphi_d} \\ \cos\theta_d/2 e^{-i\varphi_d} \end{pmatrix}_1 \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 \\
& +\sin^2\theta/2 e^{i\varphi} \begin{pmatrix} \sin\theta_d/2 e^{-i\varphi_d} \\ \cos\theta_d/2 e^{-i\varphi_d} \end{pmatrix}_1 \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2 .
\end{aligned}
$$
(118)

The $M^{\widehat{(\mathbf{a})}} = 0$ State   For this case, the probability amplitude is [1]

$$
\begin{aligned}
\Psi(1,0^{\widehat{(\mathbf{a})}};(m_1)_u^{\widehat{(\mathbf{c}_1)}},(m_2)_v^{\widehat{(\mathbf{c}_2)}}) &= \sum_{\alpha,\alpha'} \{ \sum_l \zeta(1,0^{\widehat{(\mathbf{a})}};1,M_l^{\widehat{(\mathbf{k})}}) \\
&\times \vartheta(1,M_l^{\widehat{(\mathbf{k})}};(m_1)_\alpha^{\widehat{(\mathbf{k})}},(m_2)_{\alpha'}^{\widehat{(\mathbf{k})}}) \} \\
&\times \Phi((m_1)_\alpha^{\widehat{(\mathbf{k})}},(m_2)_{\alpha'}^{\widehat{(\mathbf{k})}};(m_1)_u^{\widehat{(\mathbf{c}_1)}},(m_2)_v^{\widehat{(\mathbf{c}_2)}}) \\
&= \sum_j \chi(1,0^{\widehat{(\mathbf{a})}};B_{\alpha\alpha'})\Phi(B_{\alpha\alpha'};(m_1)_u^{\widehat{(\mathbf{c}_1)}},(m_2)_v^{\widehat{(\mathbf{c}_2)}}),
\end{aligned}
$$
(119)

where

$$
\chi(1,0^{\widehat{(\mathbf{a})}};B_{\alpha\alpha'}) = \sum_l \zeta(1,0^{\widehat{(\mathbf{a})}};1,M_l^{\widehat{(\mathbf{k})}})\vartheta(1,M_l^{\widehat{(\mathbf{k})}};B_{\alpha\alpha'}).
$$
(120)

The ζ's change because they are functions of the initial-state quantum numbers. Thus, we have [9]

$$
\zeta(1,0^{\widehat{(\mathbf{a})}};1,1^{\widehat{(\mathbf{k})}}) = -\frac{1}{\sqrt{2}}\sin\theta e^{-i\varphi}
$$
(121)

$$
\zeta(1,0^{\widehat{(\mathbf{a})}};1,0^{\widehat{(\mathbf{k})}}) = \cos\theta
$$
(122)

and

$$
\zeta(1,0^{\widehat{(\mathbf{a})}};1,(-1)^{\widehat{(\mathbf{k})}}) = \frac{1}{\sqrt{2}}\sin\theta e^{i\varphi}.
$$
(123)

However, the ϑ's do not change. Thus, using Eqs. (98) - (100) for the ϑ's, we find that

$$
\begin{aligned}
\chi(1,0^{\widehat{(\mathbf{a})}};&(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = \\
&\zeta(1,0^{\widehat{(\mathbf{a})}};1,1^{\widehat{(\mathbf{k})}})\vartheta(1,1^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) \\
&+\zeta(1,0^{\widehat{(\mathbf{a})}};1,0^{\widehat{(\mathbf{k})}})\vartheta(1,0^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) \\
&+\zeta(1,0^{\widehat{(\mathbf{a})}};1,(-1)^{\widehat{(\mathbf{k})}})\vartheta(1,(-1)^{\widehat{(\mathbf{k})}};(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}},(+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) \\
&= -\frac{1}{\sqrt{2}}\sin\theta e^{-i\varphi}.
\end{aligned}
$$
(124)

The other χ's are found to be

$$\chi(1,0^{\widehat{(a)}};(+\tfrac{1}{2})^{\widehat{(k)}},(-\tfrac{1}{2})^{\widehat{(k)}}) \quad = \quad \chi(1,0^{\widehat{(a)}};(-\tfrac{1}{2})^{\widehat{(k)}},(+\tfrac{1}{2})^{\widehat{(k)}})$$

$$= \quad \frac{1}{\sqrt{2}}\cos\theta \tag{125}$$

and

$$\chi(1,0^{\widehat{(a)}};(-\tfrac{1}{2})^{\widehat{(k)}},(-\tfrac{1}{2})^{\widehat{(k)}}) = \frac{1}{\sqrt{2}}\sin\theta e^{i\varphi}. \tag{126}$$

Thus

$$\begin{aligned}
[\Psi(1,0^{\widehat{(a)}};(m_1)^{\widehat{(c_1)}},(m_2)^{\widehat{(c_2)}})] = \quad & -\frac{1}{\sqrt{2}}\sin\theta e^{-i\varphi}\begin{pmatrix}\cos\theta_d/2\\-\sin\theta_d/2\end{pmatrix}_1\begin{pmatrix}\cos\theta_f/2\\-\sin\theta_f/2\end{pmatrix}_2 \\
& +\frac{1}{\sqrt{2}}\cos\theta\begin{pmatrix}\cos\theta_d/2\\-\sin\theta_d/2\end{pmatrix}_1\begin{pmatrix}\sin\theta_f/2e^{-i\varphi_f}\\\cos\theta_f/2e^{-i\varphi_f}\end{pmatrix}_2 \\
& +\frac{1}{\sqrt{2}}\cos\theta\begin{pmatrix}\sin\theta_d/2e^{-i\varphi_d}\\\cos\theta_d/2e^{-i\varphi_d}\end{pmatrix}_1\begin{pmatrix}\cos\theta_f/2\\-\sin\theta_f/2\end{pmatrix}_2 \\
& +\frac{1}{\sqrt{2}}\sin\theta e^{i\varphi}\begin{pmatrix}\sin\theta_d/2e^{-i\varphi_d}\\\cos\theta_d/2e^{-i\varphi_d}\end{pmatrix}_1\begin{pmatrix}\sin\theta_f/2e^{-i\varphi_f}\\\cos\theta_f/2e^{-i\varphi_f}\end{pmatrix}_2.
\end{aligned} \tag{127}$$

The $M^{\widehat{(a)}} = -1$ State For this case, the probability amplitude is

$$\begin{aligned}
\Psi(1,(-1)^{\widehat{(a)}};(m_1)_u^{\widehat{(c_1)}},(m_2)_v^{\widehat{(c_2)}}) &= \sum_{\alpha,\alpha'}\{\sum_l \zeta(1,(-1)^{\widehat{(a)}};1,M_l^{\widehat{(k)}}) \\
&\times \vartheta(1,M_l^{\widehat{(k)}};(m_1)_\alpha^{\widehat{(k)}},(m_2)_{\alpha'}^{\widehat{(k)}})\} \\
&\times \Phi((m_1)_\alpha^{\widehat{(k)}},(m_2)_{\alpha'}^{\widehat{(k)}};(m_1)_u^{\widehat{(c_1)}},(m_2)_v^{\widehat{(c_2)}}) \\
&= \sum_{\alpha,\alpha'}\chi(1,(-1)^{\widehat{(a)}};B_{\alpha\alpha'})\Phi(B_{\alpha\alpha'};(m_1)_u^{\widehat{(c_1)}},(m_2)_v^{\widehat{(c_2)}}),
\end{aligned} \tag{128}$$

where

$$\chi(1,(-1)^{\widehat{(a)}};B_{\alpha\alpha'}) = \chi(1,(-1)^{\widehat{(a)}};B_{\alpha\alpha'}) := \sum_l \zeta(1,(-1)^{\widehat{(a)}};1,M_l^{\widehat{(k)}})\vartheta(1,M_l^{\widehat{(k)}};B_{\alpha\alpha'}) \tag{129}$$

The ζ's for this case are

$$\zeta(1,(-1)^{\widehat{(a)}};1,1^{\widehat{(k)}}) = -\sin^2\theta/2e^{-i\varphi}, \tag{130}$$

$$\zeta(1,(-1)^{\widehat{(a)}};1,0^{\widehat{(k)}}) = \frac{1}{\sqrt{2}}\sin\theta \tag{131}$$

and

$$\zeta(1,(-1)^{\widehat{(a)}};1,(-1)^{\widehat{(k)}}) = -\cos^2\frac{\theta}{2}e^{i\varphi}. \tag{132}$$

Since the ϑ's remain the same, it follows that

$$\chi(1,(-1)^{\widehat{(a)}};(+\tfrac{1}{2})^{\widehat{(k)}},(+\tfrac{1}{2})^{\widehat{(k)}})) = -\sin^2\theta/2e^{-i\varphi}, \tag{133}$$

$$\chi(1, (-1)^{\widehat{(\mathbf{a})}}; (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})) = \chi(1, (-1)^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}))$$

$$= \frac{1}{2}\sin\theta, \tag{134}$$

and

$$\chi(1, (-1)^{\widehat{(\mathbf{a})}}; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})) = -\cos^2\theta/2 e^{i\varphi}. \tag{135}$$

As a result, we obtain for the matrix state

$$
\begin{aligned}
[\Psi(1, (-1)^{\widehat{(\mathbf{a})}}; (m_1)^{\widehat{(\mathbf{c_1})}}, (m_2)^{\widehat{(\mathbf{c_2})}})] = & -\sin^2\theta/2 e^{-i\varphi}\begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1 \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 \\
& +\frac{1}{2}\sin\theta \begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1 \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2 \\
& +\frac{1}{2}\sin\theta \begin{pmatrix} \sin\theta_d/2 e^{-i\varphi_d} \\ \cos\theta_d/2 e^{-i\varphi_d} \end{pmatrix}_1 \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 \\
& -\cos^2\theta/2 e^{i\varphi}\begin{pmatrix} \sin\theta_d/2 e^{-i\varphi_d} \\ \cos\theta_d/2 e^{-i\varphi_d} \end{pmatrix}_1 \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2.
\end{aligned}
\tag{136}
$$

5.2.2 The Singlet State

Having obtained the triplet states, we now seek the singlet state. The general formula is Eq. (67). The generalized probability amplitude for the singlet state is

$$\Psi(s=0, M=0; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}) = \Psi(0,0; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}})$$

$$= \sum_{\alpha,\alpha'} \chi(0,0; (m_1)_\alpha^{\widehat{(\mathbf{k})}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{k})}})\Phi((m_1)_\alpha^{\widehat{(\mathbf{k})}}, (m_2)_{\alpha'}^{\widehat{(\mathbf{k})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}). \tag{137}$$

The χ's are now directly Clebsch-Gordan coefficients for the case of total spin 0 and subspins $1/2$ and $1/2$. Thus,

$$\chi(0,0; (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}0, \tfrac{1}{2}\tfrac{1}{2}0) = 0, \tag{138}$$

$$\chi(0,0; (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}0, \tfrac{1}{2}, -\tfrac{1}{2}0) = \frac{1}{\sqrt{2}}, \tag{139}$$

$$\chi(0,0; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}0, -\tfrac{1}{2}\tfrac{1}{2}0) = -\frac{1}{\sqrt{2}} \tag{140}$$

and

$$\chi(0,0; (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}) = C(\tfrac{1}{2}\tfrac{1}{2}0, -\tfrac{1}{2}, -\tfrac{1}{2}0) = 0. \tag{141}$$

This means that the generalized probability amplitude is

$$\Psi(0,0; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}) = \tfrac{1}{\sqrt{2}}\Phi((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}})$$

$$-\tfrac{1}{\sqrt{2}}\Phi((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}}, (+\tfrac{1}{2})^{\widehat{(\mathbf{k})}}; (m_1)_u^{\widehat{(\mathbf{c_1})}}, (m_2)_v^{\widehat{(\mathbf{c_2})}}). \tag{142}$$

Hence, the matrix state is

$$[\Psi(0,0; (m_1)^{\widehat{(\mathbf{c_1})}}, (m_2)^{\widehat{(\mathbf{c_2})}})] = \tfrac{1}{\sqrt{2}}[\eta_1((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2$$

$$-\tfrac{1}{\sqrt{2}}[\eta_1((-\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_1[\eta_2((+\tfrac{1}{2})^{\widehat{(\mathbf{k})}})]_2. \tag{143}$$

Therefore, the generalized matrix form for the singlet state is

$$[\Psi(0,0;(m_1)^{(\widehat{\mathbf{c}_1})},(m_2)^{(\widehat{\mathbf{c}_2})})] = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1 \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2$$
$$-\frac{1}{\sqrt{2}}\begin{pmatrix} \sin\theta_d/2 e^{-i\varphi_d} \\ \cos\theta_d/2 e^{-i\varphi_d} \end{pmatrix}_1 \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 . \tag{144}$$

6 Recovery of Standard Results

It is now easy to see how the standard results come about from the current ones. First of all, if $\widehat{\mathbf{a}}$ is along the z axis, so that $\theta = \varphi = 0$, the triplet states become.

$$[\Psi(1,1^{(\widehat{\mathbf{a}})};(m_1)^{(\widehat{\mathbf{c}_1})},(m_2)^{(\widehat{\mathbf{c}_2})})] = \begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1 \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 , \tag{145}$$

$$[\Psi(1,0^{(\widehat{\mathbf{a}})};(m_1)^{(\widehat{\mathbf{c}_1})},(m_2)^{(\widehat{\mathbf{c}_2})})] = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1 \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2$$
$$+\frac{1}{\sqrt{2}}\begin{pmatrix} \sin\theta_d/2 e^{-i\varphi_d} \\ \cos\theta_d/2 e^{-i\varphi_d} \end{pmatrix}_1 \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 \tag{146}$$

and

$$[\Psi(1,(-1)^{(\widehat{\mathbf{a}})};(m_1)^{(\widehat{\mathbf{c}_1})},(m_2)^{(\widehat{\mathbf{c}_2})})] = -\begin{pmatrix} \sin\theta_d/2 e^{-i\varphi_d} \\ \cos\theta_d/2 e^{-i\varphi_d} \end{pmatrix}_1 \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2 , \tag{147}$$

while the singlet state remains

$$[\Psi(0,0;(m_1)^{(\widehat{\mathbf{c}_1})},(m_2)^{(\widehat{\mathbf{c}_2})})] = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta_d/2 \\ -\sin\theta_d/2 \end{pmatrix}_1 \begin{pmatrix} \sin\theta_f/2 e^{-i\varphi_f} \\ \cos\theta_f/2 e^{-i\varphi_f} \end{pmatrix}_2$$
$$-\frac{1}{\sqrt{2}}\begin{pmatrix} \sin\theta_d/2 e^{-i\varphi_d} \\ \cos\theta_d/2 e^{-i\varphi_d} \end{pmatrix}_1 \begin{pmatrix} \cos\theta_f/2 \\ -\sin\theta_f/2 \end{pmatrix}_2 . \tag{148}$$

The operator, Eqs. (73) - (81), remains unchanged.

We recover the standard formulas if in addition, $\widehat{\mathbf{d}} = \widehat{\mathbf{f}} = \widehat{\mathbf{k}}$:

$$[\Psi(1,1^{(\widehat{\mathbf{a}})};(m_1)^{(\widehat{\mathbf{c}_1})},(m_2)^{(\widehat{\mathbf{c}_2})})] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 , \tag{149}$$

$$[\Psi(1,0^{(\widehat{\mathbf{a}})};(m_1)^{(\widehat{\mathbf{c}_1})},(m_2)^{(\widehat{\mathbf{c}_2})})] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2$$
$$+\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2 , \tag{150}$$

$$[\Psi(1,(-1)^{(\widehat{\mathbf{a}})};(m_1)^{(\widehat{\mathbf{c}_1})},(m_2)^{(\widehat{\mathbf{c}_2})})] = -\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 \tag{151}$$

and

$$[\Psi(0,0;(m_1)^{\widehat{(\mathbf{c_1})}},(m_2)^{\widehat{(\mathbf{c_2})}})] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}_2 - \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_2.$$

(152)

In this limit, the elements of $[r^{(1)}]_1$ are

$$r_{11}^{(1)} = \cos^2\theta_1/2 r^{(1)}((+\tfrac{1}{2})^{\widehat{(\mathbf{c_1})}}) - \sin^2\theta_1/2 r^{(1)}((-\tfrac{1}{2})^{\widehat{(\mathbf{c_1})}}),$$

(153)

$$r_{12}^{(1)} = \frac{1}{2}\sin\theta_1 e^{-i\varphi_1}(r^{(1)}((+\tfrac{1}{2})^{\widehat{(\mathbf{c_1})}}) - r^{(1)}((-\tfrac{1}{2})^{\widehat{(\mathbf{c_1})}})),$$

(154)

$$r_{21}^{(1)} = r_{12}^{(1)*}$$

(155)

and

$$r_{22}^{(1)} = -r_{11}^{(1)}.$$

(156)

The elements of $[r^{(2)}]_2$ are

$$r_{11}^{(2)} = \cos^2\theta_2/2 r^{(2)}((+\tfrac{1}{2})^{\widehat{(\mathbf{c_2})}}) - \sin^2\theta_2/2 r^{(2)}((-\tfrac{1}{2})^{\widehat{(\mathbf{c_2})}})$$

(157)

$$r_{12}^{(2)} = \frac{1}{2}\sin\theta_2 e^{-i\varphi_2}(r^{(2)}((+\tfrac{1}{2})^{\widehat{(\mathbf{c_2})}}) - r^{(2)}((-\tfrac{1}{2})^{\widehat{(\mathbf{c_2})}}))$$

(158)

$$r_{21}^{(2)} = r_{12}^{(2)*}$$

(159)

$$r_{22}^{(2)} = -r_{11}^{(2)}$$

(160)

In the event that the quantities r_1 and r_2 are spin projections, we may assign the values $+1$ if the projections are up with respect to the respective unit vectors $\widehat{\mathbf{c}}_1$ and $\widehat{\mathbf{c}}_2$ and -1 if they are down with respect to these vectors. Thus, $r^{(1)}((\pm\tfrac{1}{2})^{\widehat{(\mathbf{c_1})}}) = \pm 1$. In that case, the generalized operator $[r^{(1)}]_1$ has the elements

$$r_{11}^{(1)} = \cos(\theta_d - \theta_1) - 2\sin\theta_d\sin\theta_1\sin^2((\varphi_d - \varphi_1)/2),$$

(161)

$$r_{12}^{(1)} = -\sin\theta_d\cos\theta_1 + \sin\theta_1\cos\theta_d\cos(\varphi_d - \varphi_1) + i\sin\theta_1\sin(\varphi_d - \varphi_1),$$

(162)

$$r_{21}^{(1)} = r_{12}^{(1)}$$

(163)

and

$$r_{22}^{(1)} = -r_{11}^{(1)}.$$

(164)

Exactly the same expressions hold for the operator $[r^{(2)}]_2$, except that the subscript d is replaced by the subscript f, and where the numeral 1 does not give the row or column of a matrix element, it is replaced by 2.

In the limit $\widehat{\mathbf{d}} = \widehat{\mathbf{f}} = \widehat{\mathbf{k}}$, the operators become

$$[r^{(1)}]_1 = \begin{pmatrix} \cos\theta_1 & \sin\theta_1 e^{-i\varphi_1} \\ \sin\theta_1 e^{i\varphi_1} & -\cos\theta_1 \end{pmatrix}_1,$$

(165)

and

$$[r^{(2)}]_2 = \begin{pmatrix} \cos\theta_2 & \sin\theta_2 e^{-i\varphi_2} \\ \sin\theta_2 e^{i\varphi_2} & -\cos\theta_2 \end{pmatrix}_2,$$

(166)

the well-known standard forms. Thus, we see that the standard results are obtained easily and logically from this approach. These states are all normalized to unity, as is easily proved. Also, they are mutually orthogonal.

22

7 Discussion and Conclusion

In this paper, we have derived the standard matrix treatment of spin addition from probability amplitudes. This confirms the fact, first brought out in Ref. [7], that spin theory can be based on probability amplitudes. It also confirms the correctness of the Landé approach to quantum mechanics.

A very important observation arising from this paper is that the standard results for spin matrix mechanics are only a special case of more generalized ones. Despite that calculations can be successfully performed with the standard quantities even in ignorance of this fact, our understanding of spin theory is incomplete until we take this fact on board. Although the results in this paper relate to spin addition, the general observation that the standard theory of angular momentum addition is not generalized enough is true. Therefore, more generalized results await the application of the current approach to the addition of spins other than those corresponding to spin-1/2 systems. By the same token, the addition of spin and orbital angular momentum will lead to more generalized results. One can extend this observation to the case of the addition of three or more spins. The elucidation of angular momentum theory cannot be regarded as complete until the task of obtaining the generalized results is finished.

From the generalized probability amplitudes derived in Ref. [1], we presented two different matrix treatments for the triplets states, and one matrix treatment for the singlet state. We found that we could express the singlet state or the triplet states in terms of 4×4 operators and vectors with four rows each. In addition, we could express the triplet states by means of 3×3 operators and vectors with three rows. In both cases, the total space was not decomposed into two spaces corresponding to the constituent subsystems 1 and 2. But in the standard treatment, the operator is the product of an operator in the space of subsystem 1 and of an operator in the space of subsystem 2. The state consists of terms which are products of vectors in the subspaces of systems 1 and 2. There is thus this difference between the standard treatment and the new treatments in Ref. [1]. This difference appears to be far from trivial. In the present generalized standard treatment, we could only succeed in deriving results by assuming that the operator of the arbitrary observable R was factorizable into factors depending on the spaces of subsystem 1 and of subsystem 2. In the new treatments of Ref. [1], this was not necessary. Thus, when we need to deal with "entangled" observables R, we need to resort to the new treatments, or to forgo matrix mechanics and use probability-amplitude mechanics.

Our work highlights the power of the Landé interpretation of quantum mechanics. This approach continues to surprise, and it is all but certain that it has new results to yield when applied to areas of quantum mechanics other than spin theory.

8 References

1. Mweene H. V., "New Treatment of Systems of Compounded Angular Momentum", quant-ph/9907082

2. Landé A., "From Dualism To Unity in Quantum Physics", Cambridge University Press, 1960.

3. Landé A., "New Foundations of Quantum Mechanics", Cambridge University Press, 1965.

4. Landé A., "Foundations of Quantum Theory," Yale University Press, 1955.

5. Landé A., "Quantum Mechanics in a New Key," Exposition Press, 1973.

6. Rose M. E., "Elementary Theory of Angular Momentum", John Wiley and Sons, Inc. (New York), 1957

7. Mweene H. V., "Derivation of Spin Vectors and Operators From First Principles", quant-ph/9905012

8. Mweene H. V., "Generalized Spin-1/2 Operators and Their Eigenvectors", quant-ph/9906002

9. Mweene H. V., "Vectors and Operators for Spin 1 Derived From First Principles", quant-ph/9906043

10. Mweene H. V., "Alternative Forms of Generalized Vectors and Operators for Spin 1/2", quant-ph/9907031

11. Mweene H. V., "Spin Description and Calculations in the Landé Interpretation of Quantum Mechanics", quant-ph/9907033